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Five or Ten New Proofs of the Pythagorean Theorem

Ne′Kiya Jackson and Calcea Johnson

Abstract. We present five trigonometric proofs of the Pythagorean theorem, and our method for finding proofs (Section 5) yields at least five more.

1. INTRODUCTION. Perhaps no subject in mathematics generates more confusion and anxiety for high school students than trigonometry. It's beyond the scope of this paper (and beyond our ability) to examine why trigonometry is so confusing, but one reason may be that there are two different ways to define the same trigonometric terms, as in Figure 2 of Section 2. Figure 1 shows how these methods are usually reconciled and yet it's possible this figure does more harm than good. Students may not realize that two competing versions of trigonometry have been stamped onto the same terminology. In that case, trying to make sense of trigonometry can be like trying to make sense of a picture where two different images have been printed on top of each other.

We believe the most sensible way to avoid this confusion is to give the procedures distinct names, reflecting the distinct ideas that underlie them. But only one of these methods is actually trigonometric, and by focusing on this genuine version (and ignoring the misnamed one) a large collection of new proofs of the Pythagorean theorem can be found.

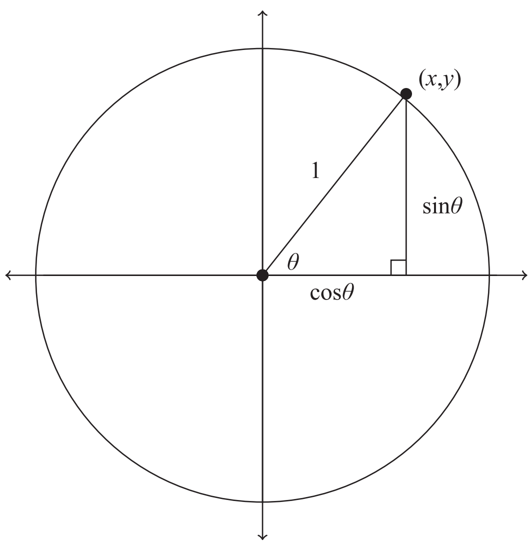

Figure 1. The most harmful figure in mathematics?

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2. WHAT IS A TRIGONOMETRIC PROOF? The word *trigonometry* is derived from the Greek words "trigonon" (triangle), and "metron"(measure), so naturally the trigonometric functions are obtained by measuring triangles. In fact, the trigonometric ratios sine and cosine are defined for an acute angle α by creating a right triangle ABC in which α is one of the two acute angles (as on the left side of Figure 2), and then comparing the lengths of two of the three sides: $\sin\alpha$ is defined as the quotient of the opposite leg BC and the hypotenuse AB, and $\cos\alpha$ is the quotient of the adjacent leg AC and the hypotenuse. These definitions are usually taught with the familiar acronym SOHCAHTOA.

But defining sine or cosine by measuring a right triangle works only for an acute angle, and all other angles—those that measure either 0° or less, or 90° or more—require an entirely different method. For these angles, we employ the unit circle instead: we start at the point $(1, 0)$ and traverse the circle in the counter-clockwise direction (clockwise for negative angles) until the desired central angle α is reached, landing us at a point (x, y). We then define $\cos\alpha = x$ and $\sin\alpha = y$.[1]

For an acute angle, these two methods give the same value for the sine or cosine function, as Figure 1 was designed to show, but only the first method can reasonably be called *trigonometric*. The second method might more appropriately be called *cyclotopic*, from the Greek words for "circle" and "location." (Figure 2).

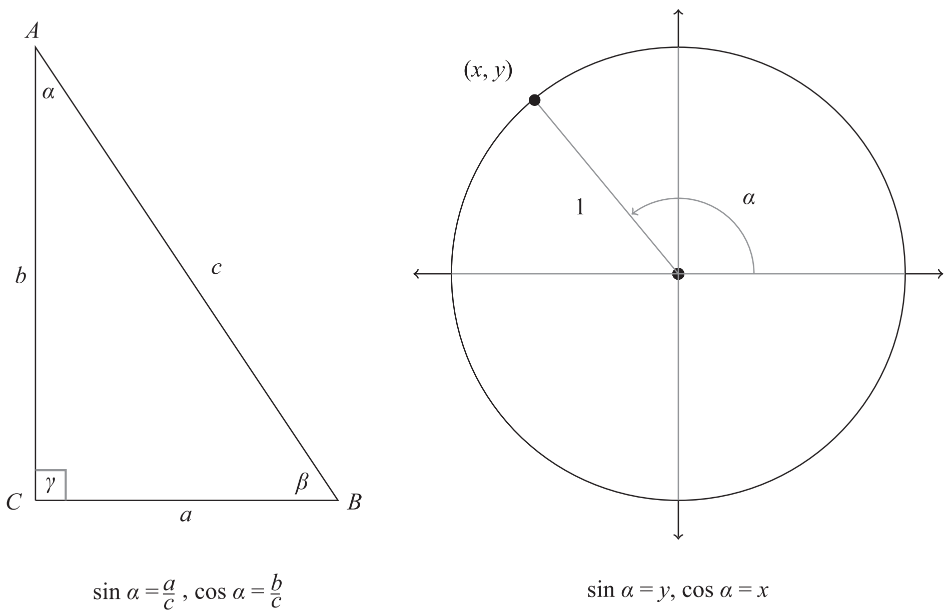

Figure 2. The trigonometric and cyclotopic definitions of sine and cosine.

In practical terms, the distinction between these methods means that proving Pythagoras's theorem via the Law of Cosines (we start with $c^2 = a^2 + b^2 - 2ab\cos\gamma$ and let γ be a right angle) is a cyclotopic proof and not a trigonometric one: trigonometry cannot compute the cosine of a right angle, whereas cyclotopic measurement tells

[1]Many people believe that all trigonometric proofs of Pythagoras's theorem are circular; see for example [**1**], which contains a large number of Pythagorean proofs but also contains this mistaken belief (p. viii, 244). The problem likely originates (or is at least reinforced) when the unit circle is used to define sine and cosine, so if students have already used the Pythagorean theorem to show the circle's equation is $x^2 + y^2 = 1$ (and thus $\sin^2\theta + \cos^2\theta = 1$) then of course a trigonometric proof would be circular.

us that $\cos(90°) = 0$. Likewise, proving the Pythagorean theorem using the formula for $\cos(\alpha - \beta)$ (let $\alpha = \beta$ in the identity $\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$) is also cyclotopic rather than trigonometric, as is the proof using the formula for $\sin(\alpha + \beta)$, where α and β are complements.

The claim that a proof is trigonometric can be denied on other grounds as well. For example, one of the best-known proofs of Pythagoras's theorem[2] uses the similarity $\triangle ABC \sim \triangle ACD \sim \triangle CBD$, as in Figure 3: since $\frac{a}{c} = \frac{x}{a}$ and $\frac{b}{c} = \frac{y}{b}$, we have $c = x + y = \frac{a^2}{c} + \frac{b^2}{c}$ so that $a^2 + b^2 = c^2$.

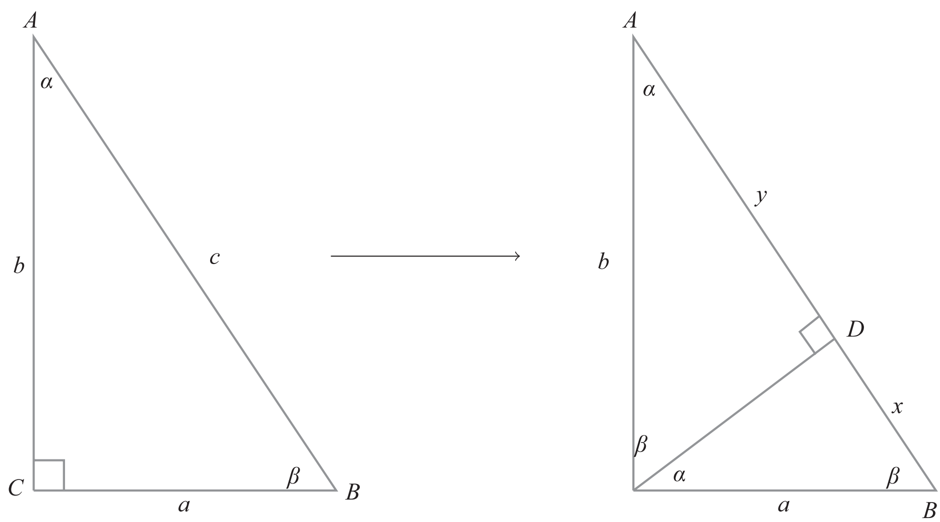

Figure 3. Proof by similar triangles.

But this proof is easily rewritten as trigonometry. Since $\frac{a}{c} = \frac{x}{a} = \sin\alpha$ we have $x = a\sin\alpha = (c\sin\alpha)\sin\alpha = c\sin^2\alpha$, and similarly $y = c\cos^2\alpha$. Then $c = x + y = c(\sin^2\alpha + \cos^2\alpha)$, from which $1 = \sin^2\alpha + \cos^2\alpha = (\frac{a}{c})^2 + (\frac{b}{c})^2$ and thus $a^2 + b^2 = c^2$. But using trigonometric terminology here adds nothing—in fact it only complicates a simpler view of the same exact approach—so we would say this proof employs similar triangles rather than trigonometry.

More generally, any proof that $a^2 + b^2 = c^2$ can be reformulated into a "trigonometric" proof simply by writing $c\sin\alpha$ for a and $c\cos\alpha$ for b (or by re-scaling the sides a, b, and c to $\sin\alpha$, $\cos\alpha$, and 1) to first prove that $\sin^2\alpha + \cos^2\alpha = 1$, after which the reverse substitutions $\sin\alpha = \frac{a}{c}$ and $\cos\alpha = \frac{b}{c}$ show that $a^2 + b^2 = c^2$. This illusion shows we need to be skeptical of a "trigonometric" proof of the Pythagorean theorem that works in this roundabout way (that is, by first proving the identity $\sin^2\alpha + \cos^2\alpha = 1$) to make sure the "trigonometry" is not just a needless restatement of side lengths using sine and cosine terminology.[3]

[2]Zorian Lučić in [2] gives a detailed account of the historical mathematical context for the interested reader to explore.

[3]Proving $a^2 + b^2 = c^2$ is not the same as proving $\sin^2\alpha + \cos^2\alpha = 1$, just as trigonometry is not the same as "cyclotopy": the former makes sense only for right triangles and their acute angles, while the latter makes sense for any angle, and doesn't even require a triangle at all. So one might be tempted to say a proof of the Pythagorean theorem must start with a figure of a right triangle and must then show directly that $a^2 + b^2 = c^2$. The hundreds of diagrams throughout [1]—one for each proof—make it clear that its author E. Loomis believed this was the only legitimate way to prove Pythagoras's theorem, which explains why he disqualified the many

In truth, we have no idea how to draw a clear line between "trigonometric" proofs of Pythagoras's theorem and non-trigonometric proofs. But with the requirements above we have a start, and by our criteria (which are less strict than the criteria of [1]) two proofs of the Pythagorean theorem qualify as trigonometric. The first belongs to J. Zimba, whose proof [3] uses the algebraic properties of the compound angle formulas to show that $\sin^2 x + \cos^2 x = 1$ for any acute angle x. The other proof ([4]) belongs to N. Luzia, who uses the compound angle formulas and the half-angle formula to show that $\sin^2 \frac{\theta}{2} + \cos^2 \frac{\theta}{2} = 1$ for any acute angle θ. Note that Luzia's method fails for the isosceles right triangle (when the angle $\frac{\theta}{2}$ in question is 45°) but works when $45 < \frac{\theta}{2} < 90$, since then $\sin^2 \frac{\theta}{2} + \cos^2 \frac{\theta}{2} = \cos^2(90 - \frac{\theta}{2}) + \sin^2(90 - \frac{\theta}{2}) = 1$.

3. PRELIMINARIES. In this section, we verify that our proofs aren't circular, i.e., that none of the theorems we use in our proofs (Section 4) have already assumed the Pythagorean theorem to be true. We note that the definitions of trigonometric functions for acute angles follow from similar triangles. These functions may be viewed as defined directly on the angles or their acute measure and are the "right triangle definitions." We freely employ other basic results from Euclidean geometry that precede the Pythagorean theorem, such as the Angle Addition postulate and the property that a perpendicular dropped from the vertex of an acute triangle meets the opposite side. We also use the measure of area of triangles and squares. This notion is generally viewed as more sophisticated than similar triangles, but it also precedes the Pythagorean theorem and is often used in proofs of the theorem. D. Clark and S. Pathania [5] provide a comprehensive reference to the geometry. For simplicity, we chose not to distinguish between angles and their degree measurements, nor between line segments and their lengths.

A. The Angle Addition Formulas. Our proofs use the Angle Addition Formulas for sine and cosine, and Figure 4 demonstrates that when α, β, and $\alpha + \beta$ are all acute angles, we have $\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$ and $\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$.

Thus if $\alpha < 45°$ with $\sin\alpha = \frac{a}{c}$ and $\cos\alpha = \frac{b}{c}$, we have $\sin(2\alpha) = 2\sin\alpha\cos\alpha = \frac{2ab}{c^2}$ and $\cos(2\alpha) = \cos^2\alpha - \sin^2\alpha = \frac{b^2-a^2}{c^2}$. And if α and β are complementary angles with $\alpha < \beta$, then since $\sin\theta = \cos(90 - \theta)$ we have $\sin(\beta - \alpha) = \cos(90 - (\beta - \alpha)) = \cos((\alpha + \beta) - (\beta - \alpha)) = \cos(2\alpha) = \frac{b^2-a^2}{c^2}$ as well.

B. The Law of Sines. Given $\triangle ABC$ in which α and β are acute angles as below, we draw the altitude CD (Figure 5):

Then $\sin\alpha = \frac{CD}{b}$ and $\sin\beta = \frac{CD}{a}$, so that $\frac{a}{\sin\alpha} = \frac{b}{\sin\beta}$. [4]

"trigonometric proofs" (called "cyclotopic" above), which would certainly have been known to someone who compiled more than 350 proofs in his lifetime. And, naturally, Loomis's claim that "There are no trigonometric proofs" of Pythagoras's theorem ([1], p.244) can be refuted only by a proof that obeys his strict requirement for Pythagorean proofs, so a proof that doesn't begin with a figure of a right triangle doesn't merit consideration.

[4] The alert reader will have noticed that the simple proofs in Sections 3A and 3B both use the "right triangle trigonometry" definitions of sine and cosine and not the cyclotopic definitions; in fact, our own informal search found no proofs of either result that use the "unit circle trigonometry" definitions. And it's nonsensical to call a method "trigonometry" if its definitions of sine and cosine are inadequate to provide simple proofs of these basic trigonometric results.

The "unit circle trigonometry" definitions of sine and cosine are equally nonsensical from an educational standpoint: imagine the confusion of a high school student who is asked to solve an actual trigonometry problem (such as determining the height of a mountain, when given some angle and distance measurements) after

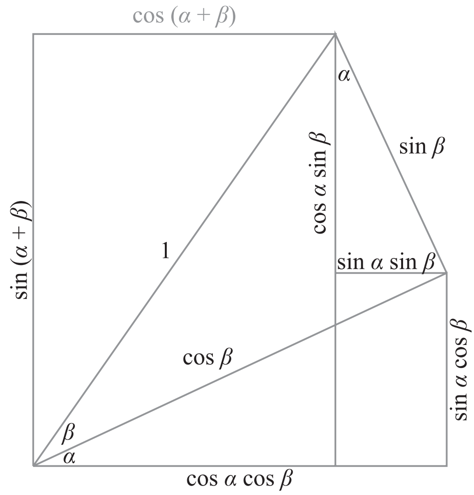

Figure 4. The Angle Addition Formulas.

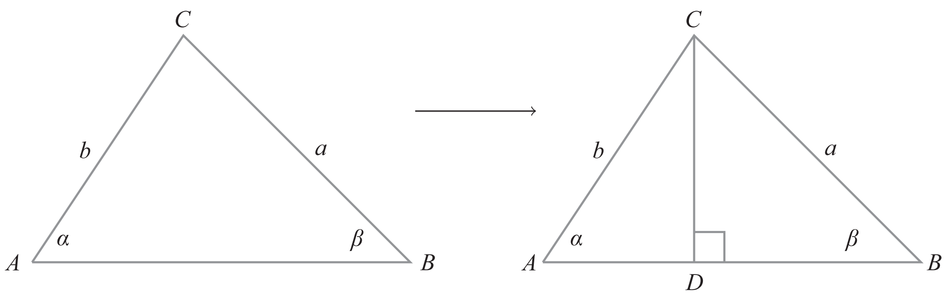

Figure 5. The Law of Sines.

C. The isosceles right triangle. The lemma in Section 5 explains why the majority of our proofs fail for the isosceles right triangle, so here we prove this special case of the Pythagorean theorem: If ABC is an isosceles right triangle ($a = b$), then two copies of ABC will create a square with side length a, while four copies create a square with side length c (Figure 6).

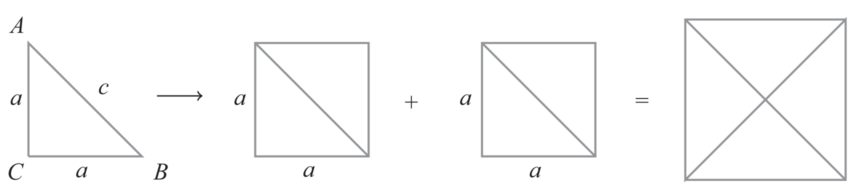

Figure 6. The Pythagorean theorem for the isosceles right triangle.

which the student draws the diagram of the observer and the mountain and then wonders where to draw the unit circle that the trigonometric functions require.

Using the unit circle only increases confusion when explaining the properties of sine and cosine as *trigonometric* functions, although it's indispensable when explaining their properties as *periodic* functions. It makes no sense to treat these two usages as if they are identical, and generations of high school students have been harmed by this mistake.

For the isosceles right triangle, then, $a^2 + b^2 = c^2$ is as simple as $2 + 2 = 4$.

4. FIVE NEW PROOFS OF THE PYTHAGOREAN THEOREM. Since we already proved the Pythagorean theorem for the isosceles right triangle, we assume in the first four of our five proofs below that ABC is a non-isosceles right triangle in which $a < b$ or equivalently, $\alpha < 45^\circ < \beta$. In accordance with the strict requirement of [1], we begin each proof with a figure of a right triangle.

A. The first proof. Our first proof begins by reflecting $\triangle ABC$ across the line $\overleftrightarrow{AC}$ through A and C to create the isosceles triangle ABB'.

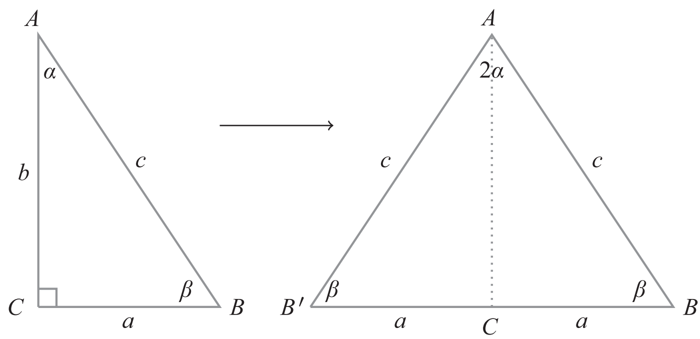

Figure 7. Creating $\triangle ABB'$.

We now construct right triangle $AB'D$ as on the left in Figure 8 by creating a right angle at vertex B' (so that $m\angle BB'D = 90 - \beta = \alpha$) and extending side AB to meet the new line segment at point D. We then fill $\triangle B'BD$ with progressively smaller and smaller scale copies of the original right triangle ABC, as at right in Figure 8.

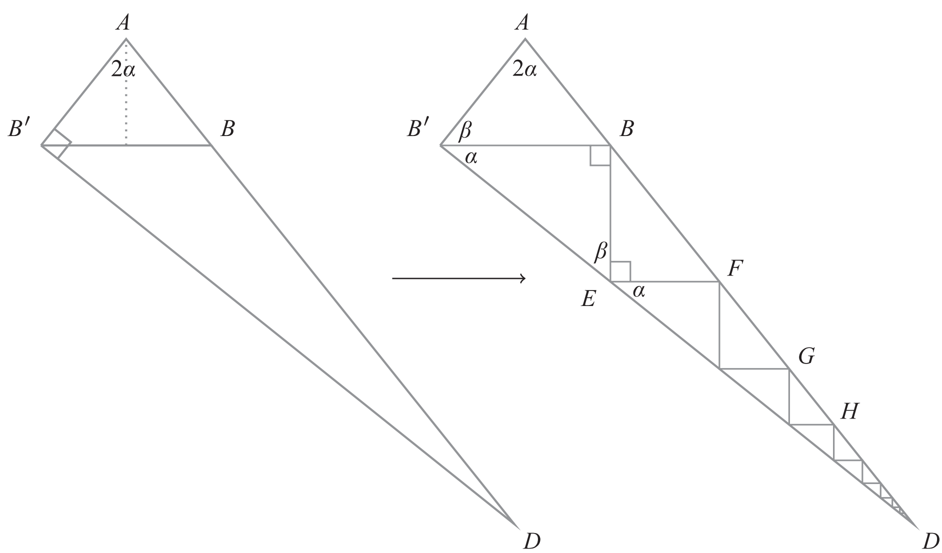

Figure 8. The first proof.

Since BB' has length $2a$ and is the longer leg of $\triangle B'EB \sim \triangle ABC$, the ratio of sides $a : b : c$ shows the shorter leg BE has length $(2a)\frac{a}{b} = \frac{2a^2}{b}$. But BE is the longer leg of $\triangle BFE$, so the hypotenuse BF of $\triangle BFE$ has length $(\frac{2a^2}{b})(\frac{c}{b}) = \frac{2a^2c}{b^2}$. By construction, the shorter leg in each triangle is also the longer leg in the next triangle, which means that successive triangles have the ratio $\frac{a}{b}$; but then alternate triangles have ratio $\frac{a^2}{b^2}$, so that $FG = (\frac{a^2}{b^2})BF = \frac{2a^4c}{b^4}$, and $GH = (\frac{a^2}{b^2})FG = \frac{2a^6c}{b^6}$, etc. Thus the hypotenuse AD of right triangle $AB'D$ has length $AB + BF + FG + GH + \cdots = c(1 + \frac{2a^2}{b^2} + \frac{2a^4}{b^4} + \frac{2a^6}{b^6} + \cdots)$.

In $\triangle AB'D$ we have $\cos(2\alpha) = \frac{AB'}{AD} = \frac{c}{AD}$ and therefore $AD = \frac{c}{\cos(2a)}$ (Figure 8).

We equate these two expressions for AD to find

$$
\begin{aligned}
& c(1 + \frac{2a^2}{b^2} + \frac{2a^4}{b^4} + \frac{2a^6}{b^6} + \cdots) = \frac{c}{\cos(2\alpha)} \\
\Rightarrow \quad & c(1 + \frac{2a^2/b^2}{1 - a^2/b^2}) = \frac{c}{\cos^2\alpha - \sin^2\alpha} \\
\Rightarrow \quad & 1 + \frac{2a^2}{b^2 - a^2} = \frac{1}{(b/c)^2 - (a/c)^2} \\
\Rightarrow \quad & \frac{b^2 + a^2}{b^2 - a^2} = \frac{c^2}{b^2 - a^2} \\
\Rightarrow \quad & a^2 + b^2 = c^2.
\end{aligned}
$$

Note that one step of our proof used the well-known Sum of Convergent Series formula $a + ar + ar^2 + ar^3 + \cdots = \frac{a}{1-r}$ to determine that

$$\frac{2a^2}{b^2} + \frac{2a^4}{b^4} + \frac{2a^6}{b^6} + \cdots = \frac{2a^2/b^2}{1 - a^2/b^2}.$$

B. The second proof. Given right triangle ABC, we locate point D on $\overleftrightarrow{BC}$ so that $m\angle BAD = \alpha$ as below, and therefore $m\angle ADC = 90 - 2\alpha = \beta - \alpha$ (Figure 9).

We first apply the Law of Sines to $\triangle ACD$:

$$
\begin{aligned}
& \frac{CD}{\sin(2\alpha)} = \frac{AC}{\sin(\beta - \alpha)} \\
\Rightarrow \quad & \frac{CD}{2ab/c^2} = \frac{b}{(b^2 - a^2)/c^2} \\
\Rightarrow \quad & CD = \frac{2ab^2}{b^2 - a^2}
\end{aligned}
$$

from which $BD = CD - BC = \frac{2ab^2}{b^2 - a^2} - a = \frac{a(a^2+b^2)}{b^2 - a^2}$.

Next we apply the Law of Sines to $\triangle ABD$:

$$\frac{BD}{\sin\alpha} = \frac{AB}{\sin(\beta - \alpha)}$$

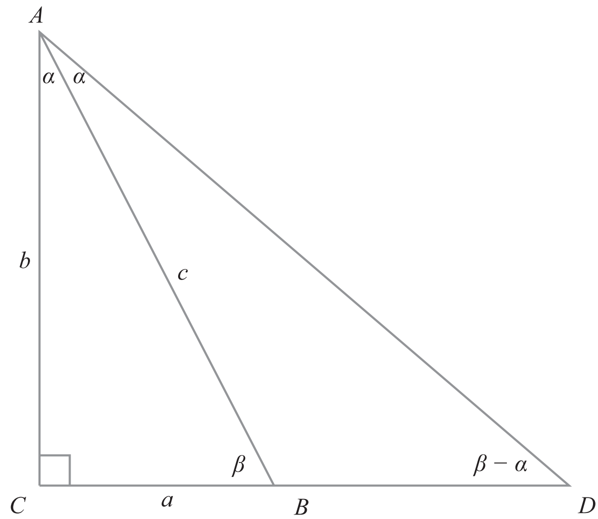

Figure 9. The second proof.

$$\Rightarrow \qquad \frac{BD}{a/c} = \frac{c}{(b^2 - a^2)/c^2}$$

$$\Rightarrow \qquad BD = \frac{ac^2}{b^2 - a^2}.$$

Comparing the two values of BD, we find $\frac{a(a^2+b^2)}{b^2-a^2} = \frac{ac^2}{b^2-a^2}$ and then $a^2 + b^2 = c^2$.

C. The third proof. We locate point D on AC so that $m\angle CBD = \beta - \alpha$, and therefore $m\angle ABD = \beta - (\beta - \alpha) = \alpha$ and $m\angle BDC = 90 - (\beta - \alpha) = 2\alpha$ (Figure 10).

By definition, $\sin(2\alpha) = \frac{BC}{BD}$ so that $BD = \frac{BC}{\sin(2\alpha)} = \frac{a}{2ab/c^2} = \frac{c^2}{2b}$, and then $CD = BD \sin(\beta - \alpha) = (\frac{c^2}{2b})(\frac{b^2-a^2}{c^2}) = \frac{b^2-a^2}{2b}$. Thus $AD = AC - CD = b - \frac{b^2-a^2}{2b} = \frac{a^2+b^2}{2b}$.

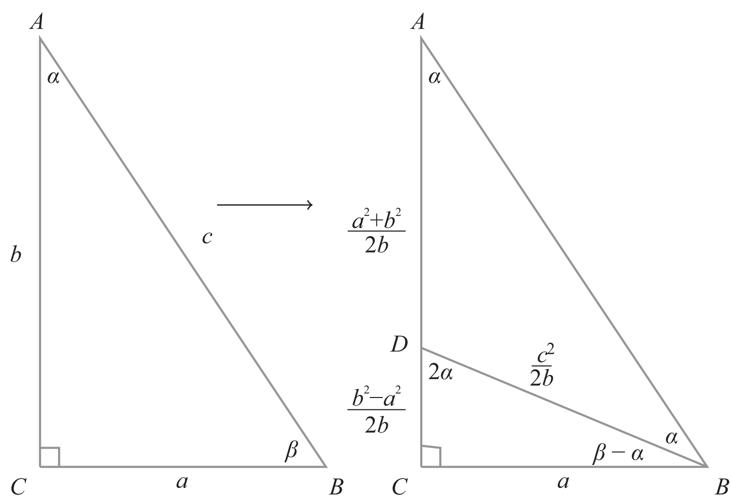

Figure 10. The third proof.

But since $\triangle ABD$ is isosceles, we have $AD = BD$ so that $\frac{a^2+b^2}{2b} = \frac{c^2}{2b}$, or $a^2 + b^2 = c^2$.

D. The fourth proof. We draw the perpendicular bisector DE of the hypotenuse AB (so that $\triangle AED \sim \triangle ABC$) and then we construct the rectangle $AOBC$ and draw its diagonals. By reflective symmetry, $m\angle BCD = m\angle CBD = \beta$; but then $m\angle DCE = 90 - \beta = \alpha$ and $m\angle BDC = 180 - (\beta + \beta) = 2\alpha$. We also have $m\angle CDE = 90 - 2\alpha = \beta - \alpha$ (Figure 11).

Since $AD = BD$, we have $AD = BD = \frac{c}{2}$, and the ratio $a : b : c$ of the sides of $\triangle AED$ shows that $DE = AD(\frac{a}{b}) = \frac{ac}{2b}$ and $AE = AD(\frac{c}{b}) = \frac{c^2}{2b}$. Thus $CE = AC - AE = b - \frac{c^2}{2b} = \frac{2b^2-c^2}{2b}$.

We apply the Law of Sines to $\triangle CDE$ to find

$$\begin{aligned} & \frac{(2b^2 - c^2)/2b}{\sin(\beta - \alpha)} = \frac{ac/2b}{\sin\alpha} \\ \Rightarrow \quad & \frac{2b^2 - c^2}{\sin(\beta - \alpha)} = \frac{ac}{\sin\alpha} \\ \Rightarrow \quad & \frac{2b^2 - c^2}{(b^2 - a^2)/c^2} = \frac{ac}{a/c} \\ \Rightarrow \quad & \frac{2b^2 - c^2}{b^2 - a^2} = 1 \\ \Rightarrow \quad & 2b^2 - c^2 = b^2 - a^2 \\ \Rightarrow \quad & a^2 + b^2 = c^2. \end{aligned}$$

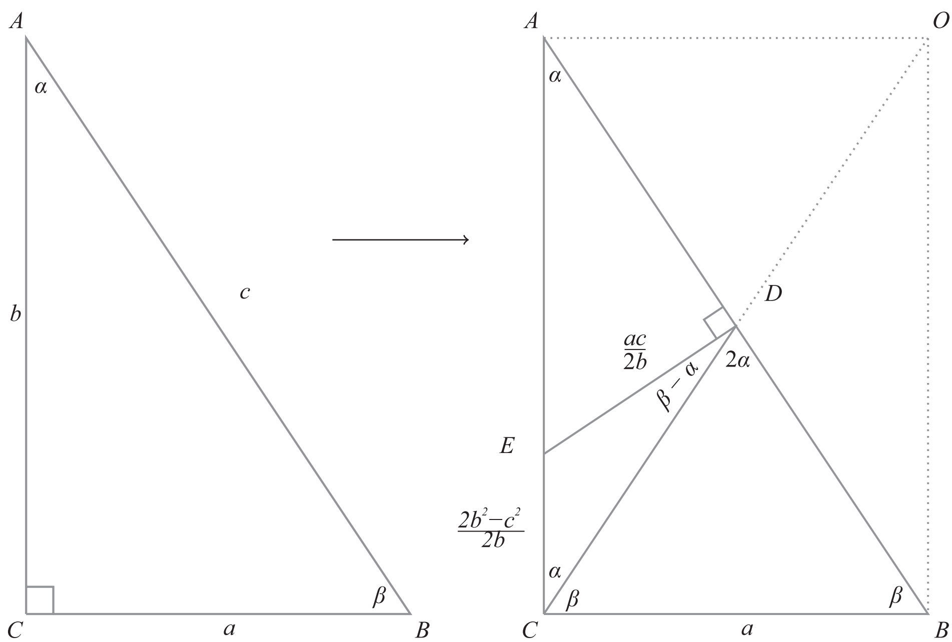

Figure 11. The fourth proof.

E. The fifth proof. Unlike our first four proofs, our fifth proof works for the isosceles right triangle. Given right triangle ABC with $\alpha \leq \beta$, for any constant k $(0 < k < 1)$ we can draw DE so that $\triangle ABC \sim \triangle ADE$ with scale factor k. We then draw DF so that $m\angle EDF = 2\alpha$, and we choose k so that F lies between B and C.

If $m\angle\alpha < 45$, then DF and EC can be extended to meet at point G, creating right triangle DEG in which $m\angle G = \beta - \alpha$. Since $DE \parallel BC$ we have $m\angle BFD = m\angle EDF = 2\alpha$, and then $m\angle BDF = 180 - (2\alpha + \beta) = \beta$. Applying the Law of Sines to $\triangle BDF$, we find $\frac{BF}{\sin\beta} = \frac{DF}{\sin\beta} = \frac{BD}{\sin(2\alpha)}$ so that $BF = DF = \frac{BD\sin\beta}{\sin(2\alpha)} = \frac{(1-k)c^2}{2a}$, and in $\triangle DEG$ we have $\sin(\beta - \alpha) = \frac{DE}{DG}$ so that $DG = \frac{DE}{\sin(\beta-\alpha)} = \frac{ka}{(b^2-a^2)/c^2} = \frac{kac^2}{b^2-a^2}$. Then $FG = DG - DF = \frac{c^2[(k+1)a^2+(k-1)b^2]}{2a(b^2-a^2)}$, and since $\sin(\beta - \alpha) = \frac{CF}{FG}$ we have $CF = FG \cdot \frac{b^2-a^2}{c^2} = \frac{(k+1)a^2+(k-1)b^2}{2a}$.

When $m\angle\alpha = 45$, we still have $BF = \frac{(1-k)c^2}{2a}$ (if M is the midpoint of BD then $BM = \frac{(1-k)c}{2}$ and $BF = \frac{c}{a} \cdot BM$) and we still have $CF = \frac{(k+1)a^2+(k-1)b^2}{2a}$ $(= ka)$, so for any right triangle ABC, we find

$$
\begin{aligned}
& BC = BF + CF \\
\Rightarrow \quad & a = \frac{(1-k)c^2}{2a} + \frac{(k+1)a^2 + (k-1)b^2}{2a} \\
\Rightarrow \quad & 2a^2 = (1-k)c^2 + (k+1)a^2 + (k-1)b^2 \\
\Rightarrow \quad & 0 = (1-k)c^2 + (k-1)a^2 + (k-1)b^2 \\
\Rightarrow \quad & a^2 + b^2 = c^2.
\end{aligned}
$$

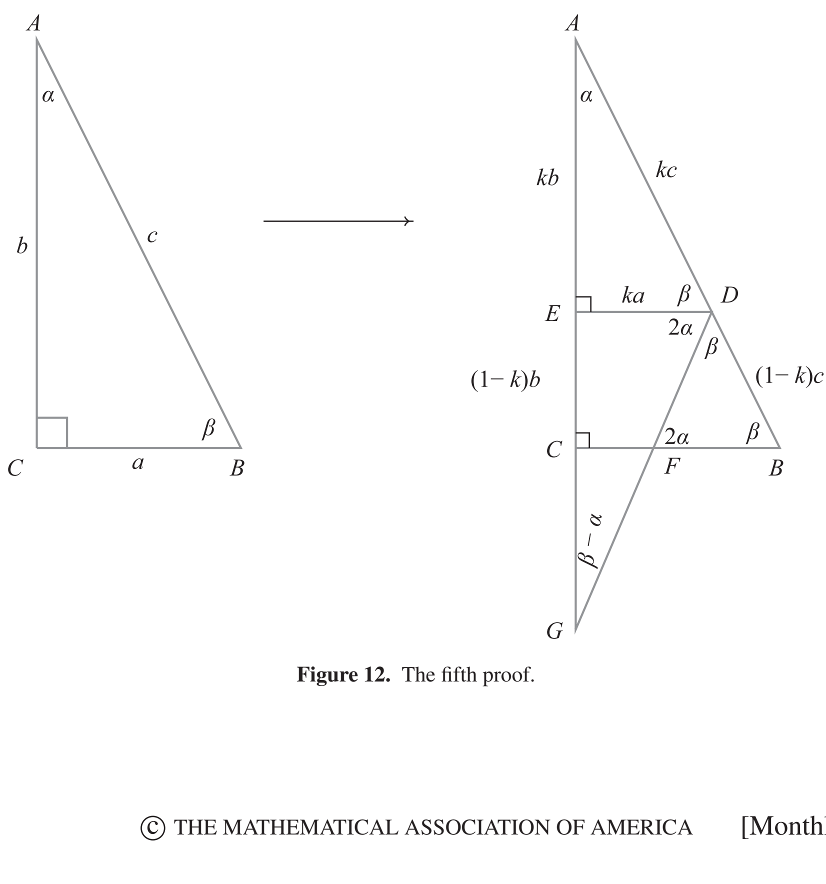

Figure 12. The fifth proof.

5. OUR METHOD. One of the fundamental questions in any creative activity is, "What can I create using what I have?" In the case of the Pythagorean theorem, the question becomes: "What right triangles can I create using the given right triangle ABC?"

This is the question we tried to answer, and we restricted our creation of new triangles to the ones whose angles are integral sums and/or differences of $\triangle ABC$'s three angles α, β, and 90 ($= \alpha + \beta$) degrees. But then the answer to our question is straightforward.

Lemma 1. *a) If ABC is an isosceles right triangle (so that $\alpha = \beta = 45$) then the only triangle whose angles are integral linear combinations of α and β is the isosceles right triangle.*

b) If $\alpha < \beta$ in right triangle ABC, then there exists a right triangle whose acute angles are 2α and $\beta - \alpha$. Furthermore, 2α and $\beta - \alpha$ are the only integral linear combinations of α and β that will form the acute angles of a right triangle for every pair $\{\alpha, \beta\}$.

Proof. a) Since all three angle measurements of isosceles triangle ABC are multiples of 45, all three angle measurements in any new triangle (whose angles are restricted to sums and/or differences of the angles of $\triangle ABC$) are still multiples of 45, and so our triangle must be an isosceles right triangle. In other words, if we start with an isosceles right triangle, then we cannot create a new triangle.

b) Now suppose that $\alpha < \beta$. If an acute angle in a newly constructed right triangle measures $m\alpha + n\beta$ $(m, n \in \mathbb{Z})$ then its complement measures $90 - (m\alpha + n\beta) = (\alpha + \beta) - (m\alpha + n\beta) = (1 - m)\alpha + (1 - n)\beta$. If the integers n and $1 - n$ are both nonzero, so that one of them (say, n) is negative, then replacing n by $|\, n \,|$ we see that one of the angles measures $m\alpha - n\beta$ where $m > n > 0$. But when α measures $\frac{90n}{m+n}$ degrees, so that its complement β measures $\frac{90m}{m+n}$, this construction gives us a triangle that has an angle of $m\alpha - n\beta = m\frac{90n}{m+n} - n\frac{90m}{m+n} = 0$. This impossibility shows we must have $n = 0$, so that one of the acute angles measures $m\alpha$ for some $m \in \mathbb{N}$.

If $m = 1$ then we simply recover our original triangle ABC. If $m = 2$ then we obtain a new right triangle whose acute angles measure 2α and $\beta - \alpha$. (Note that $2\alpha < 90$ since $\alpha < 45$.) Finally, we see that $m \geq 3$ is impossible since no such triangle can exist if $30 \leq \alpha < 45$. ■

Our lemma told us exactly how to look for proofs of the Pythagorean theorem (for non-isosceles right triangles): starting with our original triangle ABC we tried to create in as many ways as possible a new right triangle whose angles measure 2α, $\beta - \alpha$, and 90 degrees.

For example, the obvious way to create an angle of 2α is to combine two copies of $\triangle ABC$, as in Figure 13.
This creates the isosceles triangle ABB' whose angles measure 2α, β, and β, and so the next step is to take one of the angles that measures β and convert it into an angle that measures either $\beta - \alpha$ or 90 degrees (Figure 13).

To create an angle of 90 degrees at vertex B', we construct a ray that makes an angle of α with BB'. If we then extend side AB to meet the ray at point D, we obtain the figure for our first proof (Figure 14).

Alternatively, if we create the angle of 2α on the other side of the hypotenuse AB and extend BC to intersect the new ray at point D, as below, we obtain the figure that leads directly to our second proof (Figure 15).

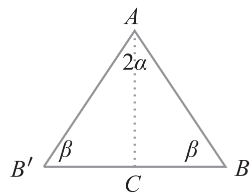

Figure 13. Creating an angle of 2α.

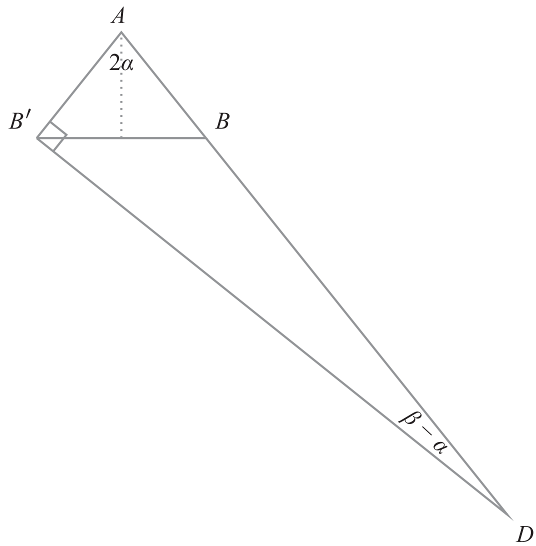

Figure 14. Creating the first proof.

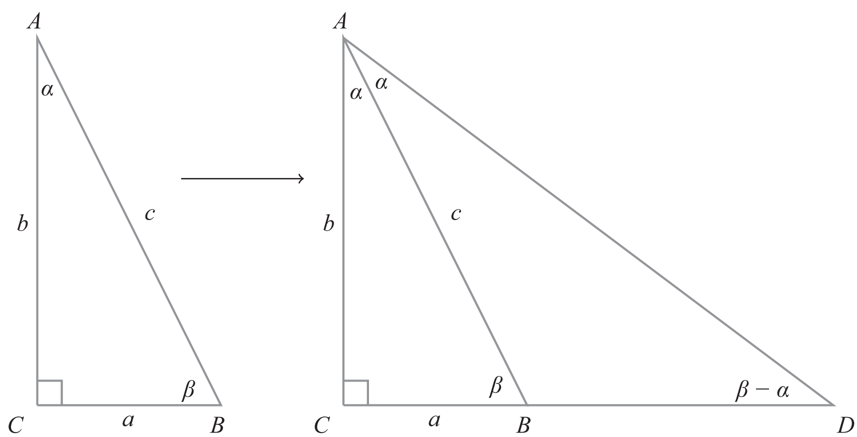

Figure 15. Creating the second proof.

This simple method produced a number of new proofs,[5] five of which are shown above while five (or more) are left for the interested reader to discover.

[5] Although our fourth proof measures an obtuse triangle (and not a right triangle) the figure for this proof was a direct result of our search.

6. CONCLUSION. The reader may be surprised to learn that the catalyst for us to start this project was a bonus question of a high school math contest. The bonus question was to create a new proof of the Pythagorean theorem. Motivated by the $500 prize, we independently decided to take on this task. It proved to be much harder than we first imagined, and we each spent many long nights trying and failing to create a proof. After roughly a month of mental labor, we each completed and submitted our work. Mr. Rich, a math volunteer at our high school, believed our proofs were novel enough to be presented at a mathematical conference. Neither of us had such confidence in our work at that point, but we decided to go along with it anyway. This is when we began to work together.

For the next two to three months, we spent all of our free time perfecting and polishing our work. We worked both independently and together after school, on weekends, and even during holidays. In the process, with Mr. Rich as our faculty advisor, we created additional proofs. We did all of this not knowing if we would even be allowed to present at the conference, which is usually only done by professional mathematicians, and occasionally college students. To our surprise, our high school work was taken seriously, and we were approved to present at the American Mathematical Society's Southeastern Sectional conference in March of 2023. Being the youngest people in the room and the youngest presenters was terrifying, but knowing that this was the culmination of all of our previous efforts gave us the confidence to present.

We were then encouraged by the AMS to submit our findings to an academic journal. This proved to be the most daunting task of all, since we had absolutely no experience writing for an academic journal. We were both also dealing with the stressors that come with adjusting to the college environment. Learning how to code in LaTeX is not so simple when you're also trying to write a 5 page essay with a group, and submit a data analysis for a lab. With the guidance and wisdom of our mentors, and a lot of personal dedication, we were able to craft this paper. The support of our family and later our community helped us to persevere. Our journey to this point was by no means simple or straightforward. There was no road map laid out for us, and there certainly was no guarantee that any of our work would go further than our own heads. There were many times when both of us wanted to abandon this project, but we decided to persevere to finish what we started.

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Editor's Note

Shortly after the authors presented these results at the AMS Sectional meeting in Atlanta, Georgia, in March 2023, my son, Colin, called them to my attention and inquired, "are you going to publish these ideas in the *Monthly*?" From that time forward, I had hoped the authors would submit their work to the *Monthly*. When they did, one of our Editorial Board members, Grant Cairns, was prepared to handle the submission. At his suggestion, we initially sent the paper back to the authors with Grant's LaTeX tips for how to make their results look better on the page. Before we sent the paper out for review, we decided that we would aim to retain the voices of the authors. Grant carefully selected the reviewers and then wrote a beautiful document for the authors that essentially served as a "guide to reading the referee reports of a mathematics paper." These young authors responded to referee reports with the finesse of longstanding professionals in the field. All this to say, I am not only honored that the authors trusted the *Monthly* with their important results but also deeply gratified by the behind-the-scenes work of the Editorial Board of the journal.

—Della Dumbaugh, Editor